

# A CHARACTERISATION OF ALGEBRAIC EXACTNESS

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**ABSTRACT.** An *algebraically exact* category is one that admits all of the limits and colimits which every variety of algebras possesses and every forgetful functor between varieties preserves, and which verifies the same interactions between these limits and colimits as hold in any variety. Such categories were studied by Adámek, Lawvere and Rosický: they characterised them as the categories with small limits and sifted colimits for which the functor taking sifted colimits is continuous. They conjectured that a complete and sifted-cocomplete category should be algebraically exact just when it is Barr-exact, finite limits commute with filtered colimits, regular epimorphisms are stable by small products, and filtered colimits distribute over small products. We prove this conjecture.

## 1. INTRODUCTION

In the series of papers [1, 3, 4] was introduced and studied the notion of an *algebraically exact* category. A category  $\mathcal{C}$  is said to be algebraically exact if, firstly, it admits all of the operations  $\mathcal{C}^{\mathcal{A}} \rightarrow \mathcal{C}$  of small arity which every variety of (finitary, many-sorted) algebras supports and every forgetful functor between varieties preserves, and secondly, it obeys all of the equations between such operations as are satisfied in every variety. Any variety admits small limits and sifted colimits, and every forgetful functor between varieties preserves them; recall from [2] that *sifted colimits* are those which commute with finite products in **Set**, most important amongst these being the filtered colimits, and the coequalisers of reflexive pairs. It follows that any algebraically exact category also admits small limits and sifted colimits; and it turns out that these two kinds of operations in fact generate all of those required of an algebraically exact category. As regarding the equations that hold between these operations, we observe that in any variety, the following four exactness properties are verified:

- (E1) Regular epimorphisms are stable under pullback, and equivalence relations are effective (i.e., the category is Barr-exact);
- (E2) Finite limits commute with filtered colimits;
- (E3) Regular epimorphisms are stable by small products;
- (E4) Filtered colimits distribute over small products.

It follows that these same conditions are verified in any algebraically exact category, and it was conjectured in [1] that, in fact, these four conditions completely characterise the algebraically exact categories amongst those categories with small limits and sifted colimits. The conjecture was proved in [4] for the case of cocomplete categories with a regular generator, and in [3] for the case

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of categories with finite coproducts; the purpose of this article is to prove it in its full generality. We shall do so using techniques developed in [7], though the arguments are straightforward enough that we can reproduce them in full here, so making this article entirely self-contained.

In order to state the conjecture more precisely, we will make use of a different description of the algebraically exact categories. We recall from [2] the construction which to every locally small category  $\mathcal{C}$  assigns its free completion  $\mathbf{Sind}(\mathcal{C})$  under sifted colimits. As in [8, Theorem 5.35], we may obtain  $\mathbf{Sind}(\mathcal{C})$  as the closure of the representables in  $[\mathcal{C}^{\text{op}}, \mathbf{Set}]$  under sifted colimits, and now the restricted Yoneda embedding  $W: \mathcal{C} \rightarrow \mathbf{Sind}(\mathcal{C})$  provides the unit at  $\mathcal{C}$  of a Kock-Zöberlein pseudomonad [10] on  $\mathbf{CAT}$ , whose pseudoalgebras are the sifted-cocomplete categories. Thus a category  $\mathcal{C}$  admits sifted colimits just when  $W: \mathcal{C} \rightarrow \mathbf{Sind}(\mathcal{C})$  admits a left adjoint.

It was shown in [1, Theorem 3.11] that if  $\mathcal{C}$  is complete, then so too is  $\mathbf{Sind}(\mathcal{C})$ ; that if  $F: \mathcal{C} \rightarrow \mathcal{D}$  is a continuous functor between complete categories, then so too is  $\mathbf{Sind}(F)$ ; and that the unit  $\mathcal{C} \rightarrow \mathbf{Sind}(\mathcal{C})$  and multiplication  $\mathbf{Sind}(\mathbf{Sind}(\mathcal{C})) \rightarrow \mathbf{Sind}(\mathcal{C})$  are always continuous functors. It follows that the pseudomonad  $\mathbf{Sind}$  restricts and corestricts to one on  $\mathbf{CONTS}$ , the 2-category of complete categories and continuous functors; and it was shown in [1, Corollary 4.4] that the pseudoalgebras for this restricted pseudomonad are precisely the algebraically exact categories described above. Thus a complete and sifted-cocomplete category  $\mathcal{C}$  is algebraically exact just when  $W: \mathcal{C} \rightarrow \mathbf{Sind}(\mathcal{C})$  admits a left adjoint which is *continuous*. For the purposes of this paper, we will take this last as our definition of an algebraically exact category; and our goal, then, is to prove:

**1.1. Theorem.** *A complete and sifted-cocomplete category  $\mathcal{C}$  is algebraically exact just when it satisfies conditions (E1)–(E4).*

In fact, as remarked above, any algebraically exact category does indeed satisfy (E1)–(E4); and so our task is to show that these conditions in turn imply algebraic exactness.

## 2. THE RESULT

The basic idea behind the proof of Theorem 1.1 is to show that any category  $\mathcal{C}$  satisfying (E1)–(E4) admits a full structure-preserving embedding into some  $\mathcal{E}$  which is an essential localisation of a presheaf topos. Any such  $\mathcal{E}$  will be algebraically exact; and now we may reflect this property along the full embedding, so concluding that  $\mathcal{C}$  itself is algebraically exact. This argument does not quite work as it stands, for reasons of size. The  $\mathcal{E}$  into which we would like to embed is a topos of sheaves on  $\mathcal{C}$ , but only when  $\mathcal{C}$  is small may such a topos be constructed; in which situation, with  $\mathcal{C}$  being small, and also small-complete, it is necessarily a preorder, which is far too restrictive. To overcome this problem, we will first prove a variant of Theorem 1.1, in which suitable bounds have been introduced on the size of the limits and colimits required, and then deduce the general result from this.

Our cardinality bounds will be governed by an infinite regular cardinal  $\kappa$ . Given any such  $\kappa$ , we define  $\kappa'$  to be the cardinal  $(\Sigma_{\gamma < \kappa} 2^\gamma)^+$ , and the pair  $(\kappa, \kappa')$  now has the property that whenever  $\mu < \kappa$  and  $\lambda < \kappa'$ , we have  $\lambda^\mu < \kappa'$ :

see [11, Proposition 2.3.5]. By a  $\kappa$ -limit we shall mean one indexed by a diagram of cardinality  $< \kappa$ , and we attach a corresponding meaning to the term  $\kappa'$ -colimit. We shall now describe a variant of the notion of algebraic exactness, which we term  $\kappa$ -*algebraic exactness*, that deals only with  $\kappa$ -limits and  $\kappa'$ -colimits.

There is a slight delicacy here as to the kinds of  $\kappa'$ -colimit we will consider. The obvious choice would be the sifted  $\kappa'$ -colimits—which we emphasise means the  $\kappa'$ -small sifted colimits, and *not* the colimits which commute in **Set** with  $\kappa'$ -small products—but this choice is in fact inappropriate. It follows from [4, Proposition 5.1] that if  $\mathcal{C}$  is complete then  $\text{Sind}(\mathcal{C})$  is the closure of the representables in  $[\mathcal{C}^{\text{op}}, \mathbf{Set}]$  under reflexive coequalisers and filtered colimits, so that a complete  $\mathcal{C}$  admits sifted colimits just when it admits reflexive coequalisers and filtered colimits. When we bound the cardinality of our colimits, it turns out to be the reflexive coequalisers together with the filtered  $\kappa'$ -colimits which are relevant, and not the sifted  $\kappa'$ -colimits; recall from [5] that the latter class of colimits is in general *strictly* larger.

We consider the 2-category  $\kappa\text{-CONTS}$  of  $\kappa$ -complete categories and  $\kappa$ -continuous functors between them; on this, we will describe a pseudomonad whose pseudoalgebras will be the  $\kappa$ -algebraically exact categories we seek to define. Observe first that as well as the pseudomonad  $\text{Sind}$  on **CAT** we also have the pseudomonad  $\mathcal{P}$  which freely adds small colimits. Proposition 4.3 and Remark 6.6 of [6] prove that if  $\mathcal{C}$  is  $\kappa$ -complete, then so is  $\mathcal{PC}$ ; that if  $F: \mathcal{C} \rightarrow \mathcal{D}$  is a  $\kappa$ -continuous functor between such categories, then so is  $\mathcal{P}F$ ; and that  $\mathcal{P}$ 's unit and multiplication are always  $\kappa$ -continuous. Thus we may restrict and corestrict  $\mathcal{P}$  to a pseudomonad on  $\kappa\text{-CONTS}$ ; and the pseudomonad of interest to us will be a submonad of this, defined as follows. For each  $\mathcal{C}$  in  $\kappa\text{-CONTS}$ , we let  $\mathcal{S}_{\kappa'}(\mathcal{C})$  denote the closure of  $\mathcal{C}$  in  $\mathcal{PC}$  under  $\kappa$ -limits, reflexive coequalisers, and filtered  $\kappa'$ -colimits, and let  $V: \mathcal{C} \rightarrow \mathcal{S}_{\kappa'}(\mathcal{C})$  denote the restricted Yoneda embedding. Now [7, Proposition 3.1] ensures that this  $V$  provides the unit at  $\mathcal{C}$  of a Kock-Zöberlein pseudomonad on  $\kappa\text{-CONTS}$ ; and a  $\kappa$ -algebraically exact category will be, by definition, a pseudoalgebra for this pseudomonad. In other words, a  $\kappa$ -complete category  $\mathcal{C}$  is  $\kappa$ -*algebraically exact* just when the embedding  $V: \mathcal{C} \rightarrow \mathcal{S}_{\kappa'}(\mathcal{C})$  admits a  $\kappa$ -continuous left adjoint. Observe that this implies that  $\mathcal{C}$  has reflexive coequalisers and filtered  $\kappa'$ -colimits, but may not imply that it has all sifted  $\kappa'$ -colimits; this is in accordance with the remarks of the preceding paragraph.

We shall now prove the following refinement of Theorem 1.1.

**2.1. Theorem.** *A category  $\mathcal{C}$  with  $\kappa$ -limits, reflexive coequalisers and filtered  $\kappa'$ -colimits is  $\kappa$ -algebraically exact just when:*

- (E1') *It is Barr-exact;*
- (E2') *Finite limits commute with filtered  $\kappa'$ -colimits;*
- (E3') *Regular epimorphisms are stable by  $\kappa$ -small products;*
- (E4') *Filtered  $\kappa'$ -colimits distribute over  $\kappa$ -small products.*

Clearly, a complete and sifted-cocomplete  $\mathcal{C}$  satisfies (E1')–(E4') for each regular  $\kappa$  if and only if it satisfies (E1)–(E4). On the other hand, we have:

**2.2. Proposition.** *A complete and sifted-cocomplete category  $\mathcal{C}$  is algebraically exact if and only if it is  $\kappa$ -algebraically exact for each regular  $\kappa$ .*

By virtue of this Proposition and the comment preceding it, we may prove Theorem 1.1 by proving Theorem 2.1, and then taking the conjunction of all its instances as  $\kappa$  ranges across the small regular cardinals.

*Proof of Proposition 2.2.* For every  $\kappa$ , we observe that  $\mathcal{S}\text{ind}(\mathcal{C})$  is closed under  $\kappa$ -limits, reflexive coequalisers and filtered  $\kappa'$ -colimits in  $[\mathcal{C}^{\text{op}}, \mathbf{Set}]$ ; whence  $\mathcal{S}_{\kappa'}(\mathcal{C}) \subset \mathcal{S}\text{ind}(\mathcal{C})$  with the inclusion preserving all  $\kappa$ -limits. Hence if  $W: \mathcal{C} \rightarrow \mathcal{S}\text{ind}(\mathcal{C})$  admits a continuous left adjoint, then by restriction each  $V: \mathcal{C} \rightarrow \mathcal{S}_{\kappa'}(\mathcal{C})$  will admit a  $\kappa$ -continuous left adjoint.

Conversely, suppose that each  $V: \mathcal{C} \rightarrow \mathcal{S}_{\kappa'}(\mathcal{C})$  admits a  $\kappa$ -continuous left adjoint. As observed above, since  $\mathcal{C}$  is complete, it follows by [4, Proposition 5.1] that  $\mathcal{S}\text{ind}(\mathcal{C})$  is the closure of the representables in  $[\mathcal{C}^{\text{op}}, \mathbf{Set}]$  under reflexive coequalisers and filtered colimits. But it is easy to see that the collection of  $\varphi \in \mathcal{S}\text{ind}(\mathcal{C})$  which lie in some  $\mathcal{S}_{\kappa'}(\mathcal{C})$  contains the representables and is closed under reflexive coequalisers and filtered colimits, and so must be all of  $\mathcal{S}\text{ind}(\mathcal{C})$ ; which is to say that  $\mathcal{S}\text{ind}(\mathcal{C}) = \bigcup_{\kappa} \mathcal{S}_{\kappa'}(\mathcal{C})$ . Thus, since each  $V: \mathcal{C} \rightarrow \mathcal{S}_{\kappa'}(\mathcal{C})$  admits a left adjoint, so too does  $W: \mathcal{C} \rightarrow \mathcal{S}\text{ind}(\mathcal{C})$ , and it remains to show that this left adjoint is continuous. Given a small diagram  $D: \mathcal{I} \rightarrow \mathcal{S}\text{ind}(\mathcal{C})$ , we may choose a regular cardinal  $\kappa$  such that  $DI \in \mathcal{S}_{\kappa'}(\mathcal{C})$  for each  $I \in \mathcal{I}$  and also  $|\mathcal{I}| < \kappa$ ; now the diagram  $D$  factors as  $D': \mathcal{I} \rightarrow \mathcal{S}_{\kappa'}(\mathcal{C})$ , and the left adjoint of  $\mathcal{C} \rightarrow \mathcal{S}_{\kappa'}(\mathcal{C})$  preserves the limit of  $D'$ : from which it follows that the left adjoint of  $W$  preserves that of  $D$ , as required.  $\square$

We now prove Theorem 2.1 for the case of a small  $\mathcal{C}$ . Given such a  $\mathcal{C}$  satisfying the conditions of the theorem, we shall embed it into a  $\kappa$ -algebraically exact category via a functor preserving  $\kappa$ -limits, reflexive coequalisers and filtered  $\kappa'$ -colimits. It will then follow that  $\mathcal{C}$  is  $\kappa$ -algebraically exact by virtue of the following result.

**2.3. Proposition.** *Let  $J: \mathcal{C} \rightarrow \mathcal{E}$  be fully faithful; suppose moreover that  $\mathcal{C}$  has, and that  $J$  preserves,  $\kappa$ -limits, reflexive coequalisers and filtered  $\kappa'$ -colimits, and that  $\mathcal{E}$  is  $\kappa$ -algebraically exact. Then  $\mathcal{C}$  is also  $\kappa$ -algebraically exact.*

*Proof.* Because  $\mathcal{E}$  is  $\kappa$ -algebraically exact, the functor  $J$  admits a left Kan extension

$$\begin{array}{ccc} \mathcal{C} & \xrightarrow{J} & \mathcal{E} \\ V \downarrow \cong & \nearrow \text{Lan}_V J & \\ \mathcal{S}_{\kappa'}(\mathcal{C}) & & \end{array}$$

along  $V$ , which may be calculated as the composite

$$\mathcal{S}_{\kappa'}(\mathcal{C}) \xrightarrow{\mathcal{S}_{\kappa'}(J)} \mathcal{S}_{\kappa'}(\mathcal{E}) \xrightarrow{L} \mathcal{E}$$

with  $L$  the  $\kappa$ -continuous left adjoint of  $V: \mathcal{E} \rightarrow \mathcal{S}_{\kappa'}(\mathcal{E})$ . Now  $\mathcal{S}_{\kappa'}(J)$  is an algebra morphism between free  $\mathcal{S}_{\kappa'}$ -algebras, and as such, preserves  $\kappa$ -limits, reflexive coequalisers and filtered  $\kappa'$ -colimits; whilst  $L$  preserves all colimits, being a left adjoint. It follows that  $\text{Lan}_V J$ , like  $J$ , preserves  $\kappa$ -limits, reflexive coequalisers and filtered  $\kappa'$ -colimits; whence the collection of  $\varphi \in \mathcal{S}_{\kappa'}(\mathcal{C})$  for which  $\text{Lan}_V J$  lands in the essential image of  $J$  contains the representables and is closed under  $\kappa$ -limits, reflexive coequalisers and filtered  $\kappa'$ -colimits, and so must be all of  $\mathcal{S}_{\kappa'}(\mathcal{C})$ . Hence  $\text{Lan}_V J$  factors through  $J$ , up-to-isomorphism;

and the factorisation  $\mathcal{S}_{\kappa'}(\mathcal{C}) \rightarrow \mathcal{C}$  so induced, which is clearly  $\kappa$ -continuous, may also be shown to be left adjoint to  $V: \mathcal{C} \rightarrow \mathcal{S}_{\kappa'}(\mathcal{C})$ , so that  $\mathcal{C}$  is indeed  $\kappa$ -algebraically exact.  $\square$

Given a small,  $\kappa$ -complete  $\mathcal{C}$ , admitting reflexive coequalisers and filtered  $\kappa'$ -colimits, and satisfying (E1')–(E4'), we now exhibit an embedding of the above form; as anticipated at the start of this section, it will in fact be an embedding into a topos. We consider the smallest topology on  $\mathcal{C}$  for which all regular epimorphisms are covering, and for which the colimit injections into each filtered  $\kappa'$ -colimit are covering. (E1') and (E2') ensure that this topology is subcanonical and so we have a full embedding  $J: \mathcal{C} \rightarrow \mathbf{Sh}(\mathcal{C})$ .

**2.4. Proposition.** *The full embedding  $J: \mathcal{C} \rightarrow \mathbf{Sh}(\mathcal{C})$  preserves  $\kappa$ -limits, reflexive coequalisers and filtered  $\kappa'$ -colimits.*

*Proof.* Clearly  $J$  preserves all limits that exist, so in particular  $\kappa$ -limits. It also preserves regular epimorphisms, since the given topology contains the regular one, and we will show below that it preserves filtered  $\kappa'$ -colimits. It will then follow that it preserves reflexive coequalisers too, since in  $\mathcal{C}$  and in  $\mathbf{Sh}(\mathcal{C})$ , we may exploit (E1') and (E2') to construct such coequalisers from finite limits, countable filtered colimits and coequalisers of equivalence relations, all of which are preserved by  $J$ ; the argument is standard and given in precisely the form we need in [3, Theorem 2.6].

It remains to show that  $J$  preserves filtered  $\kappa'$ -colimits. Observe that if  $(p_k: Dk \rightarrow X \mid k \in \mathcal{K})$  is such a colimit in  $\mathcal{C}$ , then  $J$  will preserve it just when every sheaf  $\mathcal{C}^{\text{op}} \rightarrow \mathbf{Set}$  sends it to a limit in  $\mathbf{Set}$ . Let  $F$  be such a sheaf. Since the family  $(p_k \mid k \in \mathcal{K})$  is covering, we may identify  $FX$  with the set of matching families for this covering. In other words, if

$$\begin{array}{ccc} D_{jk} & \xrightarrow{d_{jk}} & Dj \\ c_{jk} \downarrow & & \downarrow p_j \\ Dk & \xrightarrow{p_k} & X \end{array}$$

is a pullback for each  $j, k \in \mathcal{K}$ , then we may identify  $FX$  with the set

$$(*) \quad \{ \vec{x} \in \prod_k FDk \mid Fd_{jk}(x_j) = Fc_{jk}(x_k) \text{ for all } j, k \in \mathcal{K} \} .$$

Under this identification, the canonical comparison map  $FX \rightarrow \lim FD$  is just the inclusion between these sets, seen as subobjects of  $\prod_k FDk$ , and so injective; it remains to show that it is also surjective. Thus we must show that each  $\vec{x} \in \lim FD$  lies in (\*), or in other words, that  $Fd_{jk}(x_j) = Fc_{jk}(x_k)$  for each such  $\vec{x}$  and each  $j, k \in J$ . To this end, we consider the category  $\mathcal{K}'$  of cospans from  $j$  to  $k$  in  $\mathcal{K}$ ; since  $\mathcal{K}$  is filtered and  $\kappa'$ -small, it follows easily that  $\mathcal{K}'$  is too. We define a functor  $E: \mathcal{K}' \rightarrow \mathcal{C}$  by sending each cospan  $f: j \rightarrow \ell \leftarrow k: g$  in  $\mathcal{K}'$  to the apex of the pullback square

$$\begin{array}{ccc} E(f, g) & \xrightarrow{u_{f,g}} & Dj \\ v_{f,g} \downarrow & & \downarrow Df \\ Dk & \xrightarrow{Dg} & D\ell \end{array}$$

in  $\mathcal{C}$ . A simple calculation shows that  $p_k.v_{f,g} = p_j.u_{f,g}$ , so that we have induced maps  $q_{f,g} := (u_{f,g}, v_{f,g}): E(f, g) \rightarrow D_{jk}$ , constituting a cocone  $q$  under  $E$  with vertex  $D_{jk}$ . We claim that this cocone is colimiting; whereupon, by the preceding part of the argument, the comparison  $FD_{jk} \rightarrow \lim FE$  induced by  $q$  will be monic, and so the family  $(Fq_{f,g} \mid (f, g) \in \mathcal{K}')$  jointly monic. Thus in order to verify that  $Fd_{jk}(x_j) = Fc_{jk}(x_k)$ , and so complete the proof, it will be enough to observe that for each  $f: j \rightarrow \ell \leftarrow k: g$  in  $\mathcal{K}'$ , we have:

$$\begin{aligned} Fq_{f,g}(Fd_{jk}(x_j)) &= Fu_{f,g}(x_j) = Fu_{f,g}(FDf(x_\ell)) \\ &= Fv_{f,g}(FDg(x_\ell)) = Fv_{f,g}(x_k) \\ &= Fq_{f,g}(Fc_{jk}(x_k)) . \end{aligned}$$

It remains to verify that  $q$  is colimiting. For this, let  $V: \mathcal{K}' \rightarrow \mathcal{K}$  denote the functor sending a  $j, k$ -cospan to its central object, and  $\iota_1: \Delta j \rightarrow V \leftarrow \Delta k: \iota_2$  the evident natural transformations. Now we have a commutative cube

$$\begin{array}{ccccc} E & \xrightarrow{u} & \Delta D_j & & \\ \downarrow v & \searrow q & \downarrow D_{\iota_1} & \Downarrow & \\ & \Delta(D_{jk}) & \xrightarrow{\Delta d_{jk}} & \Delta(D_j) & \\ & \downarrow D_{\iota_2} & & \downarrow \Delta p_j & \\ \Delta(D_k) & \xrightarrow{\Delta c_{jk}} & DV & \searrow pV & \\ & \downarrow \Delta c_{jk} & & \downarrow & \\ & \Delta(D_k) & \xrightarrow{\Delta p_k} & \Delta X & \end{array}$$

in  $[\mathcal{K}', \mathcal{C}]$ ; its front and rear faces are pullbacks, and by (E2') will remain so on applying the functor  $\text{colim}: [\mathcal{K}', \mathcal{C}] \rightarrow \mathcal{C}$ . To show that  $q$  is colimiting is equally to show that it is inverted by  $\text{colim}$ ; for which, by the previous sentence, it is enough to show that  $pV$  is likewise inverted. But  $\mathcal{K}$ 's filteredness implies easily that  $V: \mathcal{K}' \rightarrow \mathcal{K}$  is a final functor, so that  $pV$ , like  $p$ , is a colimiting cocone, and so inverted by  $\text{colim}$  as required.  $\square$

We thus have a full structure-preserving embedding  $\mathcal{C} \rightarrow \mathbf{Sh}(\mathcal{C})$  and the only thing left to verify is that  $\mathbf{Sh}(\mathcal{C})$  is in fact  $\kappa$ -algebraically exact. The key to doing so is the following proposition.

**2.5. Proposition.** *If  $\mathcal{E}$  is reflective in a presheaf category via a  $\kappa$ -continuous reflector, then  $\mathcal{E}$  is  $\kappa$ -algebraically exact.*

*Proof.* If  $\mathcal{C}$  is small, then  $\mathcal{PC} = [\mathcal{C}^{\text{op}}, \mathbf{Set}]$ , and now the restricted Yoneda embedding  $\mathcal{PC} \rightarrow \mathcal{PPC}$  admits a continuous left adjoint  $\mathcal{PPC} \rightarrow \mathcal{PC}$ , this being the multiplication at  $\mathcal{C}$  of the pseudomonad  $\mathcal{P}$ . Since  $\mathcal{S}_{\kappa'}(\mathcal{PC})$  is closed in  $\mathcal{PPC}$  under  $\kappa$ -limits, it follows by restriction that  $\mathcal{PC} \rightarrow \mathcal{S}_{\kappa'}(\mathcal{PC})$  admits a  $\kappa$ -continuous left adjoint; and so every presheaf category is  $\kappa$ -algebraically exact. Now if  $\mathcal{E}$  is reflective in the  $\kappa$ -algebraically exact  $[\mathcal{C}^{\text{op}}, \mathbf{Set}]$  via a  $\kappa$ -continuous reflector, then it is an adjoint retract of  $[\mathcal{C}^{\text{op}}, \mathbf{Set}]$  in  $\kappa\text{-CONTS}$ , and so by a standard property of Kock-Zöberlein pseudomonads, must itself be  $\kappa$ -algebraically exact.  $\square$

Thus it is enough to show that  $\mathbf{Sh}(\mathcal{C})$  is reflective in  $[\mathcal{C}^{\text{op}}, \mathbf{Set}]$  via a  $\kappa$ -continuous reflector. This will be a consequence of the following result, which may be found proven—though with “small” harmlessly replacing our “ $\kappa$ -small”—in [9, Theorem 4.2]; we shall not recall the details, since we shall not need them in what follows.

**2.6. Proposition.** *A left exact reflector  $L: [\mathcal{C}^{\text{op}}, \mathbf{Set}] \rightarrow \mathcal{E}$  preserves all  $\kappa$ -small limits if and only if the covering sieves for the corresponding topology are closed under  $\kappa$ -small intersections in  $[\mathcal{C}^{\text{op}}, \mathbf{Set}]$ .*

We are therefore required to show that any  $\kappa$ -small intersection of covering sieves for the above-defined topology on  $\mathcal{C}$  is again covering. Clearly it is sufficient to consider the case where the sieves participating in the intersection are generating ones for the topology. We can decompose any such intersection of sieves as an intersection

$$\bigcap_{i \in I} \mathcal{S}_i \cap \bigcap_{j \in J} \mathcal{T}_j$$

where each indexing set  $I$  and  $J$  is  $\kappa$ -small, each sieve  $\mathcal{S}_i$  is generated by a regular epimorphism  $e_i: A_i \twoheadrightarrow X$  and each sieve  $\mathcal{T}_j$  is generated by a  $\kappa'$ -small filtered colimit cocone  $((q_j)_x: D_j(x) \rightarrow X \mid x \in \mathcal{A}_j)$ .

Now we can form the  $\kappa$ -small product  $\Pi_i e_i: \Pi_i A_i \rightarrow \Pi_i X$ ; by condition (E3') this is a regular epimorphism in  $\mathcal{C}$ , and by regularity, so also is its pullback  $e: A \rightarrow X$  along the diagonal  $X \rightarrow \Pi_i X$ . Clearly a map  $Z \rightarrow X$  factors through  $e$  just when it factors through each  $e_i$ , and so the covering sieve  $\mathcal{S}$  generated by  $e$  is the intersection  $\bigcap_i \mathcal{S}_i$ .

In a similar manner, we can form the filtered category  $\Pi_j \mathcal{A}_j$ ; since  $|J| < \kappa$ , and each  $|\mathcal{A}_j| < \kappa'$ , we have also that  $|\Pi_j \mathcal{A}_j| < \kappa'$ . Now on considering the diagram  $D: \Pi_j \mathcal{A}_j \rightarrow \mathcal{C}$  defined by  $D(x_j \mid j \in J) = \Pi_j D_j(x_j)$ , condition (E4') asserts that  $\Pi_j X$  is a colimit for it; so that on pulling back along the diagonal  $X \rightarrow \Pi_j X$ , we conclude that  $X$  is a colimit for the diagram  $D': \Pi_j \mathcal{A}_j \rightarrow \mathcal{C}$  which sends  $(x_j \mid j \in J)$  to the fibre product of the maps  $(q_j)_{x_j}: D_j(x_j) \rightarrow X$ . Now we see as before that the covering sieve  $\mathcal{T}$  generated by this filtered  $\kappa'$ -colimit cocone is precisely  $\bigcap_j \mathcal{T}_j$ .

It follows that  $\bigcap_i \mathcal{S}_i \cap \bigcap_j \mathcal{T}_j = \mathcal{S} \cap \mathcal{T}$  is a covering sieve, since covering sieves are always closed under finite intersections, and this completes the proof of:

**2.7. Proposition.** *If the small,  $\kappa$ -complete  $\mathcal{C}$  with reflexive coequalisers and filtered  $\kappa'$ -colimits satisfies (E1')–(E4'), then it admits a full structure-preserving embedding into a  $\kappa$ -algebraically exact category, and so is itself  $\kappa$ -algebraically exact.*

It remains to prove Theorem 2.1 for categories of no matter what size. So let  $\mathcal{C}$  be a category with  $\kappa$ -limits, reflexive coequalisers and filtered  $\kappa'$ -colimits, satisfying (E1')–(E4'). We call a full, replete subcategory  $\kappa$ -closed if it is closed in  $\mathcal{C}$  under the limits and colimits just mentioned. Clearly, each small,  $\kappa$ -closed subcategory of  $\mathcal{C}$  satisfies (E1')–(E4'), and so by the preceding proposition is  $\kappa$ -algebraically exact. We may now conclude that the same is true of  $\mathcal{C}$  by way of the following result.

**2.8. Proposition.** *A  $\kappa$ -complete  $\mathcal{C}$  admitting reflexive coequalisers and filtered  $\kappa'$ -colimits is  $\kappa$ -algebraically exact so long as all of its small  $\kappa$ -closed subcategories are.*

*Proof.* Suppose that each  $\kappa$ -closed subcategory of  $\mathcal{C}$  is  $\kappa$ -algebraically exact; we must show that  $\mathcal{C}$  is too, or in other words, that  $V: \mathcal{C} \rightarrow \mathcal{S}_{\kappa'}(\mathcal{C})$  admits a  $\kappa$ -continuous left adjoint. To this end, consider the collection of  $\varphi \in \mathcal{S}_{\kappa'}(\mathcal{C})$  for which there exists a small  $\kappa$ -closed  $J: \mathcal{D} \hookrightarrow \mathcal{C}$  with  $\varphi$  lying in the essential image of the fully faithful  $\mathcal{S}_{\kappa'}(J): \mathcal{S}_{\kappa'}(\mathcal{D}) \rightarrow \mathcal{S}_{\kappa'}(\mathcal{C})$ . It is easy to show that this collection contains the representables and is closed under  $\kappa$ -limits, reflexive coequalisers and filtered  $\kappa'$ -colimits, and so is all of  $\mathcal{S}_{\kappa'}(\mathcal{C})$ . It follows that  $\mathcal{C} \rightarrow \mathcal{S}_{\kappa'}(\mathcal{C})$  admits a left adjoint, since each  $\mathcal{D} \rightarrow \mathcal{S}_{\kappa'}(\mathcal{D})$  does by assumption.

To show that this left adjoint is moreover  $\kappa$ -continuous, consider a  $\kappa$ -small diagram  $X: \mathcal{I} \rightarrow \mathcal{S}_{\kappa'}(\mathcal{C})$ . For each  $I \in \mathcal{I}$  we can find a small  $\kappa$ -closed  $\mathcal{D}_I \subset \mathcal{C}$  with  $XI$  in the essential image of  $\mathcal{S}_{\kappa'}(\mathcal{D}_I) \rightarrow \mathcal{S}_{\kappa'}(\mathcal{C})$ ; now taking  $\mathcal{D}$  to be the closure of  $\bigcup_I \mathcal{D}_I$  in  $\mathcal{C}$  under  $\kappa$ -limits, reflexive coequalisers and filtered  $\kappa'$ -colimits, we obtain another small  $\kappa$ -closed subcategory. The diagram  $X$  factors up-to-isomorphism through the fully faithful  $\mathcal{S}_{\kappa'}(\mathcal{D}) \rightarrow \mathcal{S}_{\kappa'}(\mathcal{C})$  as  $X': \mathcal{I} \rightarrow \mathcal{S}_{\kappa'}(\mathcal{D})$ , say; and now by assumption, the left adjoint of  $\mathcal{D} \rightarrow \mathcal{S}_{\kappa'}(\mathcal{D})$  preserves the limit of  $X'$ , whence the left adjoint of  $\mathcal{C} \rightarrow \mathcal{S}_{\kappa'}(\mathcal{C})$  preserves that of  $X$ , as required.  $\square$

This completes the proof of Theorem 2.1 for categories of any size; and now, as discussed previously, taking the conjunction of all instances of this theorem as  $\kappa$  ranges over the small regular cardinals completes the proof of Theorem 1.1.

## REFERENCES

- [1] ADÁMEK, J., LAWVERE, F. W., AND ROSICKÝ, J. How algebraic is algebra? *Theory and Applications of Categories* 8 (2001), 253–283.
- [2] ADÁMEK, J., AND ROSICKÝ, J. On sifted colimits and generalized varieties. *Theory and Applications of Categories* 8 (2001), 33–53.
- [3] ADÁMEK, J., AND ROSICKÝ, J. Toward a characterization of algebraic exactness. *Journal of Algebra* 272, 2 (2004), 730–738.
- [4] ADÁMEK, J., ROSICKÝ, J., AND VITALE, E. M. On algebraically exact categories and essential localizations of varieties. *Journal of Algebra* 244, 2 (2001), 450–477.
- [5] ADÁMEK, J., ROSICKÝ, J., AND VITALE, E. M. What are sifted colimits? *Theory and Applications of Categories* 23 (2010), 251–260.
- [6] DAY, B., AND LACK, S. Limits of small functors. *Journal of Pure and Applied Algebra* 210, 3 (2007), 651–663.
- [7] GARNER, R., AND LACK, S. Lex colimits. Preprint, [arXiv:1107.0778v1](https://arxiv.org/abs/1107.0778v1), 2011.
- [8] KELLY, G. M. *Basic concepts of enriched category theory*, vol. 64 of *London Mathematical Society Lecture Note Series*. Cambridge University Press, 1982.
- [9] KELLY, G. M., AND LAWVERE, F. W. On the complete lattice of essential localizations. *Bulletin de la Société Mathématique de Belgique Série A* 41, 2 (1989), 289–319.
- [10] KOCK, A. Monads for which structures are adjoint to units. *Journal of Pure and Applied Algebra* 104, 1 (1995), 41–59.
- [11] MAKKAÏ, M., AND PARÉ, R. *Accessible categories: the foundations of categorical model theory*, vol. 104 of *Contemporary Mathematics*. American Mathematical Society, 1989.

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